Note:

- If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

A $\Rightarrow$ B
implies
is equivalent
to
Contrapositive of $A \Rightarrow B$. $\rightarrow$ not B $\Rightarrow$ not A

- Contrapositive:
If $\lim_{k\to\infty} a_k \neq 0$ or DNE, then $\sum_{k=1}^{\infty} a_k$ diverges.

- The converse $(B \Rightarrow A)$ is false.

If $\lim_{k\to\infty} a_k = 0$, it does not necessarily mean that $\sum_{k=1}^{\infty} a_k$ converges.

(example: harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$
but $\lim_{k\to\infty} \frac{1}{k} = 0$.)

Eventual convergence. If $\sum_{k=1}^{\infty} b_k$ is a series and for some large number $M \in \mathbb{N}$, $\sum_{k=M}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} b_k$ converges. [If a series eventually converges then it converges. The converse is also true!].

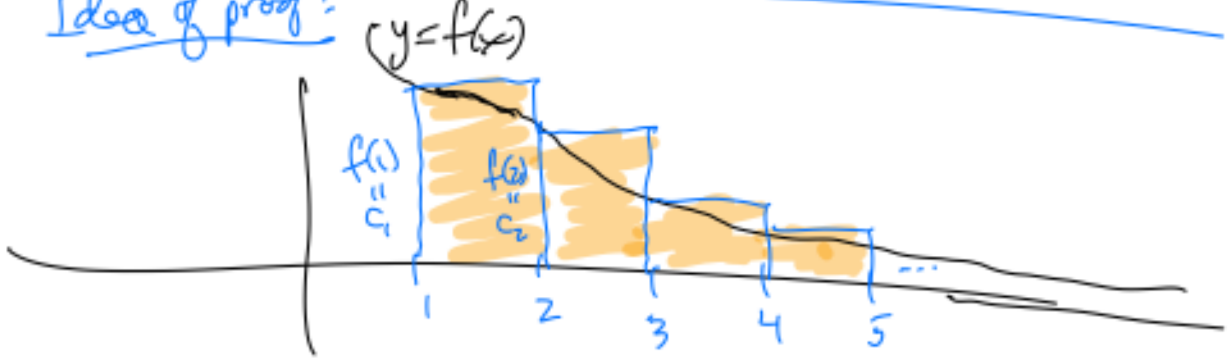
Integral Test: If $c_k \geq 0$ for $k \geq 1$, and if $c_k = f(k)$ for some decreasing continuous function f, with $f(x) \geq 0$ for $x \geq 1$, then:

$$\sum_{k=1}^{\infty} c_k \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges.}$$

"if and only if"

Thus $$\sum_{k=1}^{\infty} c_k \text{ diverges} \iff \int_1^{\infty} f(x)\,dx \text{ diverges.}$$

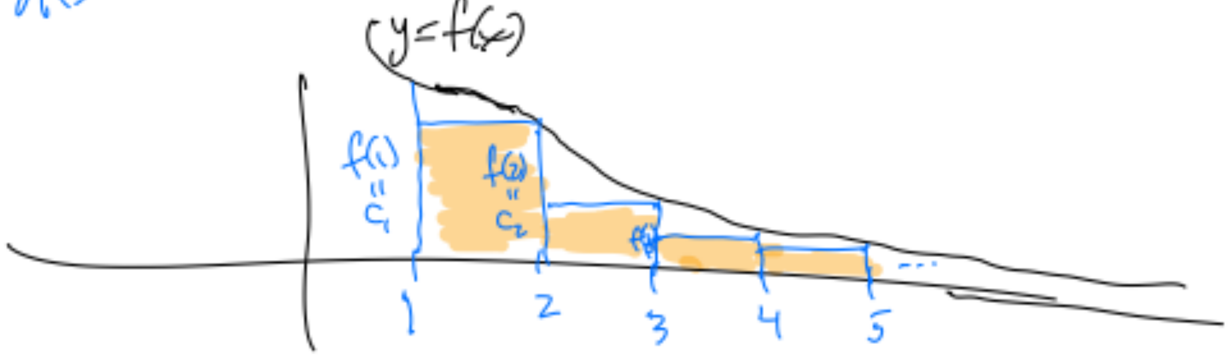
Idea of proof:



$$\text{Area of Rectangles} \geq \int_1^{\infty} f(x)\,dx$$

$$\sum_{k=1}^{\infty} \underset{c_k}{f(k)} \geq \int_1^{\infty} f(x)\,dx \geq 0$$

Also



Here

$$\left(\begin{array}{c}\text{Area of}\\ \text{rectangles}\end{array}\right) \leq \int_1^\infty f(x)\,dx$$

$$\sum_{k=2}^{\infty} f(k) \leq \int_1^\infty f(x)\,dx.$$

Putting this together

$$0 \leq \sum_{k=2}^{\infty} f(k) \leq \int_1^\infty f(x)\,dx \leq \sum_{k=1}^{\infty} f(k)$$

Also

$$0 \leq \int_2^\infty f(x)\,dx \leq \sum_{k=2}^{\infty} f(k) \leq \int_1^\infty f(x)\,dx$$

Therefore the sum $\sum_{k=1}^{\infty} f(k)$ converges

iff $\int_1^\infty f(x)\,dx$ converges.

Hypothesis: $f(x) \geq 0$ $\forall x \geq 1$, $f(x)$ is decreasing.

(Examples)

① Does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+k^2}$ converge?

Answer ① observe that $f(x) = \dfrac{1}{1+x^2} \geq 0$ for $x \geq 1$ and it's decreasing.

Next,
$$\int_1^{\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \arctan(x)\Big|_1^{b}$$

$$= \lim_{b\to\infty} \Big(\underbrace{\arctan(b)}_{\frac{\pi}{2}} - \underbrace{\arctan(1)}_{\frac{\pi}{4}}\Big) = \frac{\pi}{4} < \infty.$$

$\therefore$ The series converges. And

$$0 \leq \int_1^{\infty} \frac{1}{1+x^2}\,dx \leq \sum_{k=1}^{\infty} \frac{1}{1+k^2} \leq \int_0^{\infty} \frac{1}{1+x^2}\,dx$$

$$0 \leq \frac{\pi}{4} \leq \sum_{k=1}^{\infty} \frac{1}{1+k^2} \leq \frac{\pi}{2}$$

(omitted)

Ex Does $\displaystyle\sum_{k=2}^{\infty} \frac{2}{k\ln(k)}$ converge or not?

Answer: Notice that

$$f(x) = \frac{2}{x\ln(x)} \geq 0 \text{ for } x \geq 2,$$

and f is decreasing.

Thus we can use the integral test:

$$\int_2^{\infty} \frac{2}{x\ln(x)}\,dx$$

$$= \lim_{b\to\infty} \int_2^{b} \frac{2}{x\ln(x)}\,dx$$

let $u = \ln(x)$
$du = \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} \frac{2}{u}\,du$$

$$= \lim_{b\to\infty} 2\ln|u| \Big|_{\ln(2)}^{\ln(b)}$$

$$= \lim_{b\to\infty} \Big(\underbrace{2\ln|\underbrace{\ln(b)}_{\to\,\infty}|}_{\text{slowly}\ \to\,\infty} - 2\ln|\ln(2)| \Big) = \infty$$

$\therefore$ the series diverge.

Important Example For what values of p does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

Solution: $p>0$. If $p \le 0$ then the terms $\frac{1}{n^p}$ would not go to 0, so the series would definitely diverge to ∞.

to test out $p>0$,

consider $f(x) = \frac{1}{x^p} \ge 0$ for $x \ge 1$, decreasing in x.

$\int_1^{\infty} \frac{1}{x^p} dx$ converges if $p>1$
diverges if $p \le 1$

$$\boxed{\therefore \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p>1, \text{ diverges if } p \le 1}$$

Called p-series

Example Does $\displaystyle\sum_{n=5}^{\infty} \frac{n}{n^3+7}$ converge?

For $n \geq 5$,

$$\cancel{0} \leq \frac{n}{n^3+7} \leq \frac{n}{n^3} = \frac{1}{n^2}$$

Since $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^2}$ converges (p-series with $p=2>1$),

By the comparison test,

$\displaystyle\sum_{n=5}^{\infty} \frac{n}{n^3+7}$ converges also.

Finer version of the Comparison Test: called the Limit Comparison Test.

If $a_k \geq 0 \ \forall k$, $b_k \geq 0 \ \forall k$
"for all k"

and $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = L \neq 0, \neq \infty$.

Then $\sum a_k$ converges $\iff$ $\sum b_k$ converge.

$\sum a_k$ diverges $\iff$ $\sum b_k$ diverges.